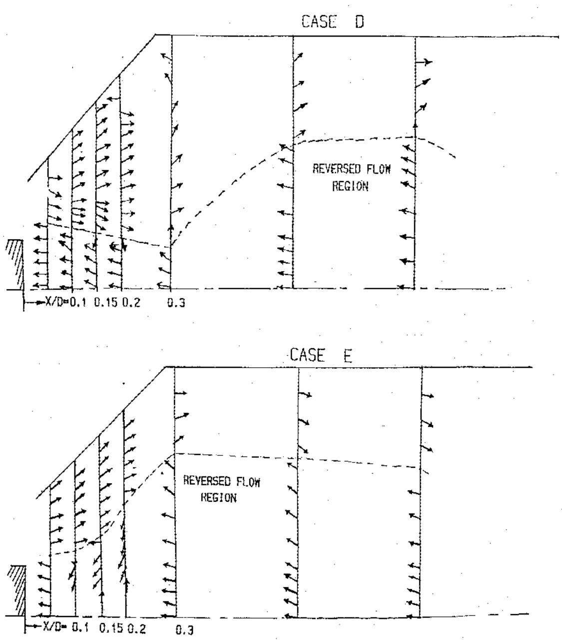

Fig. 3 Flow directions on the symmetric plane of cases D and E.

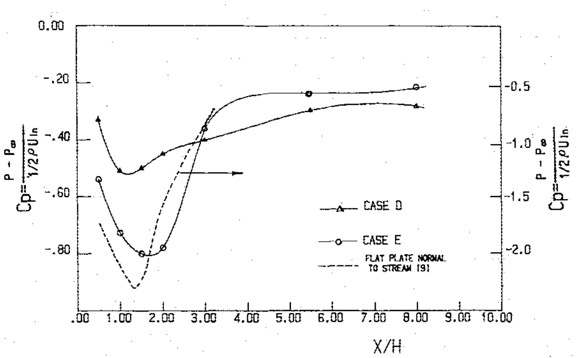

Fig. 4 Comparison of the pressure coefficient distribution along the central axis of cases D and E and the case of flow behind a flat plate normal to the stream.

dicate the reversed flow regions. It is obvious that case D has two cells, with the first smaller cell oriented downward toward the central axis. Due to swirling vortex breakdown, the second cell is parallel to the axis. Thus, these two cells stabilize each other and form a bottleneck-like shape of recirculation. Complicated unsteady flow is observed between the first and second cells. Surprisingly, case E also possesses two cells in the recirculation zone. However, unlike case D, the first smaller cell is oriented upward away from the central axis due to adverse pressure force. Flow between these two cells is complicated and unsteady. The distribution of center line pressure coefficient $C_p = (p - p_\infty)/(\rho U_{in}^2/2)$ of cases D and E, together with that of the flow behind a flat plate normal to stream,[10] are plotted in Fig. 4. The typical wake low pressure "trough" of the nonswirl wake bubble has also been noticed for both cases. The minimum static pressure is reached in $1 < x/H < 2$, and then the pressure is recovered in the "reattaching" region. Finally, the pressure reaches the plateau and the first recirculation is completed at $x/H = 3.0$, which corresponds to $x/D = 0.3$, the onset of the second cell in Fig. 4. Behind the plateau, the pressure shows a completely different trend, indicating different flow characteristics. This also confirms the conjecture of two-cell structure of the wake bubble and central vortex breakdown recirculation made above.

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Solutions of One-Dimensional Steady Nozzle Flow Revisited

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Introduction

THE differential equations for one-dimensional steady nozzle flows are easily integrated to give equations connecting any two states of the flow considered. Despite the simplicity of their forms, little has been elaborated on the solutions of the equations in the literature.[1-5] In the discussion of the well-known pressure/Mach number distribution curves corresponding to varying back pressure in a divergent or convergent-divergent nozzle (Fig. 1), the proper parameters needed, and their functional relationship, are often obscured because it was not clear how the curve was constructed. Heretofore, to find loss of total pressure (and entropy increase) across a shock wave in a nozzle, one employed a trial and error procedure in which the shock wave location was iterated until the calculated exit pressure agreed with the specified value; then the change in total pressure immediately followed. Although the procedure is straightforward, it is nonetheless tedious and requires knowing the area distribution of the nozzle. Alternatively, a transcendental equation was solved via iteration or tabulation [pp. 211–212, Eq. (4.105) and Example 4.8 in Ref. 5]. The effect of the variables involved, however, is often buried.

In this paper we show a formulation which eliminates any iteration process for obtaining the entropy increase in the nozzle and leads to simple solution of a quadratic equation. Consequently, the proper parameters are explicitly seen in the equation and their effects on the solution are easily determined. Moreover, since only one root of the equation is physically admissible, it follows that the entropy production is uniquely determined. Hence, the shock wave is uniquely determined by this set of parameters. This point is particularly

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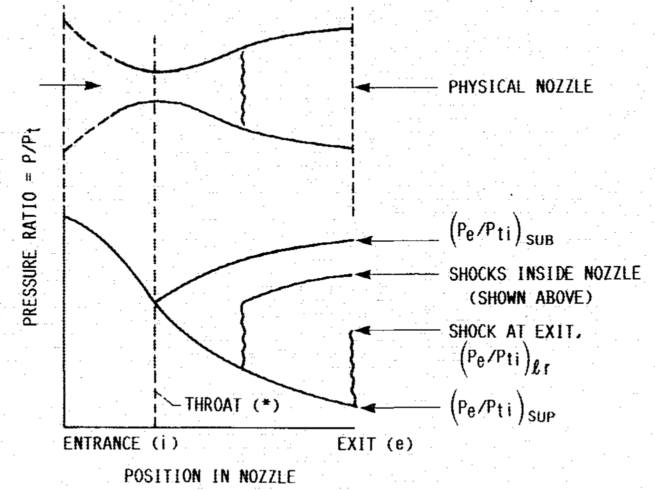

Fig. 1 Schematic distribution of pressure ratio in a convergent-divergent nozzle. Limits of exit pressure ratio are shown.

pertinent to the CFD community with regard to the proper choice of numerical boundary conditions for this type of flow.

The algebraic equation for solving isentropic exit pressures has $(\gamma+1)$ as the highest power. We prove that there are at most two real roots, although $(\gamma+1)$ is noninteger and greater than 2. Conditions under which the well-known subsonic or supersonic states exist are also given.

Formulation

The conservation of mass in a nozzle yields

$$APM/\sqrt{T} = \text{const} \tag{1}$$

where A, P, T, and M are the area, pressure, temperature, and Mach number, respectively. For homenergic (constant total enthalpy) flow, $T_{t1} = T_{t2}$, substitution of P and T in terms of the stagnation conditions (denoted by subscript t) and M gives

$$AP_tM\left(1+\frac{\gamma-1}{2}M^2\right)^{-(\gamma+1)/2(\gamma-1)} = \text{const} \tag{2}$$

Relating to the sonic condition (denoted by superscript *) isentropically, so that $P_{t1} = P_{t2}$, one obtains

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{(\gamma+1)/2(\gamma-1)} \tag{3}$$

Equation (3) now is rewritten in terms of pressure ratio by eliminating M

$$\frac{A^*}{A} = \frac{1}{\alpha}\left(1-\frac{P}{P_t}^{(\gamma-1)/\gamma}\right)^{1/2}\left(\frac{P}{P_t}\right)^{1/\gamma}$$

$$\alpha^2 = \frac{\gamma-1}{2}\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)} \tag{4}$$

It is stressed that Eqs. (3) and (4) are valid only for isentropic flows. However, entropy production is allowed across a shock wave in the homenergic (which is adiabatic) system considered, whereas the flow remains isentropic upstream and downstream of the shock wave. From Eq. (3), the states immediately upstream and downstream of the shock wave are related by

$$\frac{A_d^*}{A_u^*} = \frac{A_s/A_u^*}{A_s/A_d^*} = \frac{M_d}{M_u}\left[\frac{1+[(\gamma-1)/2]\,M_u^2}{1+[(\gamma-1)/2]\,M_d^2}\right]^{(\gamma+1)/2(\gamma-1)} \tag{5}$$

where subscripts s, u, and d indicate conditions at a shock wave and immediately upstream and downstream. Also, the shock relations in terms of the preshock Mach number M_u are

$$M_d^2 = \frac{1+[(\gamma-1)/2]\,M_u^2}{\gamma M_u^2-[(\gamma-1)/2]} \tag{6a}$$

$$\frac{P_{td}}{P_{tu}} = \left[1+\frac{2\gamma}{\gamma+1}\left(M_u^2-1\right)\right]^{-1/(\gamma-1)} \times\left[\frac{(\gamma+1)\,M_u^2}{2+(\gamma-1)\,M_u^2}\right]^{\gamma/(\gamma-1)} \tag{6b}$$

Eliminating M_d in Eq. (5) and combining Eq. (6b), we find

$$A_d^*/A_u^* = P_{tu}/P_{td} \tag{7}$$

as given in Ref. 1. Let the entrance and exit conditions be denoted by subscripts i and e. Since $A_e^* = A_d^*$, $A_i^* = A_u^*$, and $P_{tu} = P_{ti}$, evaluation of Eq. (4) at the nozzle exit gives

$$[1-(P_e/P_{td})^{(\gamma-1)/\gamma}]^{1/2}(P_e/P_{ti})\,(P_{ti}/P_{td})^{1/\gamma} = \alpha\,(P_{ti}/P_{td})(A_i^*/A_e) \tag{8}$$

Let

$$\xi = (P_e/P_{td})^{(\gamma-1)/\gamma} \tag{9}$$

and

$$c = \alpha^2\,(P_e/P_{ti})^{-2}(A_e/A_i^*)^{-2} \tag{10}$$

then Eq. (8) reduces to a quadratic equation

$$c\xi^2+\xi-1=0 \tag{11}$$

Now, since $\xi > 0$ and $c > 0$, the only admissible solution is

$$\xi = (-1+\sqrt{1+4c})/(2c) \tag{12}$$

Remark 1

With ξ solved, the loss of total pressure is found by $P_{td}/P_{tu} = (\xi)^{-\gamma/(\gamma-1)}\cdot(P_e/P_{tu})$. The entropy increase follows immediately. It is the explicit function of only the following parameters in c of Eq. (10): 1) back pressure ratio P_e/P_{tu}, 2) ratio of specific heats γ, and 3) area ratio A_e/A_i^*, but independent of area distribution insofar as this solution is concerned, despite the fact that the actual area distribution will dictate if a stable shock wave or even multiple shock waves exists. Furthermore, allowing a stable shock wave in a convergent-divergent nozzle, the sonic condition must be attained at the throat, i.e., $A_i^* = A_{th}$. It follows that nothing needs to be stated in the convergent section of the nozzle, including the inflow Mach number, i.e., independent of what happens upstream of the sonic throat. However, in a divergent nozzle where the physical sonic throat is not present, the inflow Mach number and the inlet area values are needed to provide $A_i^* = A_i(A_i^*/A_i)$, where A_i^*/A_i is function of M_i from Eq. (3). If the supersonic inlet flow is thought of as being extended isentropically to an imaginary throat (sonic state), this is then consistent with the case of the convergent-divergent nozzle.

Remark 2

Since only one root is physically admissible and the shock wave is the only mechanism producing the entropy in the problem concerned, the shock location is uniquely determined by Eq. (12) at a given $A(x)$ if only a single shock wave is allowed.

Remark 3

Now we find the limits for specifying the ratio P_e/P_{ti} of c in Eq. (10) with a shock wave in a nozzle. For a fully isentropic flow, the exit pressure is obtained by solving the equation [from Eq. (4)]

$$(P/P_{ti})^{2/\gamma}\,[1-(P/P_{ti})^{(\gamma-1)/\gamma}] = \alpha^2\,(A_e/A_i^*)^{-2} \tag{13}$$

For $A_e > A_i^*$, only two real roots exist (see Appendix), one of which corresponds to subsonic speed and is denoted as $(P_e/P_{ti})_{\rm sub}$, the other of which corresponds to supersonic speed and is denoted as $(P_e/P_{ti})_{\rm sup}$ (Fig. 1). The lower limit, $(P_e/P_{ti})_{\ell r}$, is then found by assuming a shock wave exists at the exit with upstream pressure equal to $(P_e/P_{ti})_{\rm sup}$. Hence, the shock relation yields the exit pressure

$$\left(\frac{P_e}{P_{ti}}\right)_{\ell r} = \frac{4\gamma}{\gamma^2-1}\left(\frac{P_e}{P_{ti}}\right)^{1/\gamma}_{\rm sup} - \frac{\gamma+1}{\gamma-1}\left(\frac{P_e}{P_{ti}}\right)_{\rm sup} \tag{14}$$

as the lower limit. And thus, the following inequalities set limits of back pressure ratio P_e/P_{ti} for a shock wave to stay in a nozzle

$$(P_e/P_{ti})_{\ell r} \le (P_e/P_{ti}) \le (P_e/P_{ti})_{\rm sub} \tag{15}$$

Remark 4

We turn now to consider the effect of the ratio A_e/A_i^* and γ. It is straightforward to derive

$$\frac{d\xi}{dc} = -\frac{(1-\sqrt{1+4c})^2}{4c^2\sqrt{1+4c}} \le 0 \tag{16a}$$

$$\frac{dc}{d(A_e/A_i^*)} = -2c\,(A_e/A_i^*)^{-1} < 0 \tag{16b}$$

and

$$\frac{dc}{d\gamma} = \frac{2c}{(\gamma-1)^2}\,\ell n\left(\frac{\gamma+1}{2}\right) \ge 0, \quad \gamma \ge 1 \tag{16c}$$

From these we conclude that:

1) For fixed P_e/P_{ti}, increasing exit area (A_e/A_i^*) leads to an increase in loss of total pressure and in entropy production, that is, the shock wave is located further downstream to match the required P_e/P_{ti} at an enlarged exit.

2) Decreasing γ results in an increase of entropy.

Remark 5

Expressing P in terms of M, Eq. (12) gives

$$M_e^2 = \frac{-1+\sqrt{1+4c}}{\gamma-1} \tag{17}$$

Conclusion

Solutions for steady, one-dimensional nozzle flows were revisited. We proved (in the Appendix) the existence of the supersonic and the subsonic branches of solutions. For a flow having a shock wave, a simple quadratic equation was derived with a coefficient containing a set of parameters. Consequently, a unique, physically admissible solution was determined. We also provided the conditions under which the solution exists and showed the effects of the parameters on the solution.

Appendix

Lemma

For $\gamma > 1$, then the equation

$$X^{\gamma+1} - X^2 + \beta^2 = 0, \quad 0 \le x < \infty, \quad \beta \text{ real} \tag{A1}$$

has at most two real solutions if

$$\beta^2 \le \left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)} - \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)} \tag{A2}$$

The solutions must lie between 0 and 1.

Proof

Let

$$y = X^{\gamma+1} - X^2 + \beta^2 \tag{A3}$$

hence,

$$y' = (\gamma+1)X^{\gamma} - 2X$$

$$y'' = \gamma(\gamma+1)X^{\gamma-1} - 2$$

It follows that $X_{\min} = [2/(\gamma+1)]^{1/(\gamma-1)}$ is the only minimum of $y(x)$, hence, the absolute minimum. Also, since

$$y' < 0, \qquad 0 < X < X_{\min}$$

$$y' > 0, \qquad X > X_{\min} \tag{A4}$$

it is obvious that for $0 \le X < \infty$, y decreases monotonically from a local maximum $y(0) = \beta^2$ toward the minimum and increases monotonically toward infinity as $x \to \infty$. Thus, for solutions to exist, it is required that

$$y(X_{\min}) \le 0$$

This leads to

$$\beta^2 \le \left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)} - \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)} \tag{A5}$$

Hence, two intersections with $y = 0$ (i.e., solutions) must exist if the inequality in Eq. (A5) holds. Furthermore, since $0 < X_{\min} < 1$, $y(0) = y(1) = \beta^2 > 0$ and from Eq. (A4), the solutions of Eq. (A1) must lie between 0 and 1. This concludes the proof.

Theorem

Let

$$X = (P_e/P_{ti})^{1/\gamma}$$

and

$$\beta^2 = \alpha^2(A_e/A_i^*)^{-2}, \quad \alpha^2 = [(\gamma-1)/2)][2/(\gamma+1)]^{(\gamma+1)/(\gamma-1)}$$

in (A1). Then solutions exist only if $A_e \ge A_i^*$.

Proof

From the above lemma, Eq. (A5) yields

$$A_e \ge A_i^*$$

hence, the conclusion follows.

Remark

1) If $A_e = A_i^*$, the only solution is

$$P_e/P_{ti} = (2/\gamma+1)^{\gamma/(\gamma-1)} = \Gamma$$

$$= 0.528, \qquad \text{if} \qquad \gamma = 1.4$$

i.e., second sonic throat is attained isentropically, but can be either via the supersonic or the subsonic branch of solutions.

2) If $A_e > A_i^*$, then from the lemma, two and only two isentropic solutions, P_e/P_{ti}, exist and are located in the intervals, (0, Γ) and (Γ, 1), corresponding, respectively, to supersonic and subsonic states.

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A Basis for the Analysis of Solid Continua Using the Integrated Force Method

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Introduction

ANALYSIS of the solid continuum involves solution of the field equations along with prescribed boundary conditions. Methods of analysis can be classified into three broad categories. In the first method, displacements are considered as the primary unknowns. Compatibility is satisfied automatically by imposing a continuity condition on displacements and their derivatives. The field equations are equilibrium equations (EE) in terms of displacements. All boundary conditions can be readily expressed in displacements. This approach is usually called the stiffness method (SM). In the stiffness formulation, both the field equations and all the boundary conditions are expressed in terms of displacements. In the second approach, stresses are treated as primary variables. Equilibrium in the field and on the boundary are satisfied by imposing Cauchy's differential constraints in stresses. Field equations are essentially St. Venant's compatibility conditions. In this formulation, a complete solution in terms of stresses is obtained, provided all boundary conditions are prescribed in terms of stresses. This approach is called the Flexibility Method (FM). This approach is not so convenient, particularly when some of the boundary conditions are in terms of displacements because boundary displacements cannot be easily converted to primary stress variables. The third approach is the mixed formulation and is intended primarily to overcome the difficulty in the flexibility method. Reissner-Hellinger and Washizu-Yu principles provide the basis for many of the mixed formulations currently available in the literature.

Thus, a fundamental question arises: "Is it possible to develop a formulation for mixed boundary-value problems such that the solution to stresses can be completely obtained without any recourse to displacements in the field or on the boundary?" The mixed boundary-value problem can be solved completely in terms of displacements, because the boundary stress conditions can be readily expressed in terms of displacements. Unfortunately, when stresses are considered as primary variables, the expressions for boundary displacements in terms of these primary variables (stresses) can not be readily determined.

In this context, a recent formulation called Integrated Force Method (IFM), originally developed for discrete systems,[1-6] is examined. A variational basis for this method was also given in Refs. 7 and 8. The novelty of this approach is the establishment of "boundary compatibility conditions" (BCC) analogous to the stress boundary conditions of Cauchy. In a series of publications concerned with the analysis of discrete structures,[1-6] it has been demonstrated that the complete solution can be obtained in terms of stresses, using the boundary compatibility conditions without any reference to the displacements. In this Note we examine the continuum analogue of the IFM to explore and identify the class of mixed boundary-value problems in stressed solid continuum, which can be handled entirely in terms of stresses.

Basic Theory of the Mixed Boundary-Value Problem

A typical two-dimensional mixed boundary-value problem of elasticity is shown in Fig. 1. On the boundary segment S_σ forces are prescribed, whereas displacements are constrained on the remaining portion of the boundary S_u. The variational principle of the integrated force method for the mixed boundary-value problem given in Ref. 2 can be restated as

$$\delta\Pi_s = \delta\Pi_s^{(1)} + \delta\Pi_s^{(2)} \tag{1}$$

where

$$\delta\Pi_s^{(1)} = \iint(\sigma_x\delta\epsilon_x + \sigma_y\delta\epsilon_y + \sigma_{xy}\delta\epsilon_{xy})\,\mathrm{d}x\,\mathrm{d}y + \delta\int_{S_u}\{\lambda_1(u-\bar{u}) + \lambda_2(v-\bar{v})\}\,\mathrm{d}\gamma \tag{2}$$

$$\delta\Pi_s^{(2)} = \iint(\epsilon_x\delta\phi_{,yy} + \epsilon_y\delta\phi_{,xx} - \epsilon_{xy}\delta\phi_{,xy})\,\mathrm{d}x\,\mathrm{d}y + \delta\int\{\mu_1(P-\overline{P}_x) + \mu_2(P-\overline{P}_y)\}\,\mathrm{d}\gamma \tag{3}$$

where u, v are the displacements, ϕ is the Airy's stress function, and x . . . etc. indicate differentiation with respect to the variable indicated. The body forces and initial strains are omitted for simplicity. The first part forms the basis of the traditional displacement formulation, with prescribed displacements on S_u and zero forces on S_σ. In this discussion we will consider only the second part $\Pi_s^{(2)}$. The stationary condition of the functional $\Pi_s^{(2)}$ yields the following equations for a rectangular domain:

Field Equation:

$$\nabla^4\phi = 0 \tag{4}$$

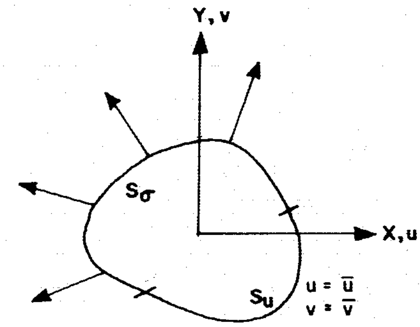

Fig. 1 A typical plane stress problem.

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